Journal of Approximation Theory **97**, 65–81 (1999)
Article ID jath.1997.3260, available online at http://www.idealibrary.com on IDEAL®

# Best Error Estimates for Discrete Abel-Gontscharoff Interpolation

Patricia J. Y. Wong

*Division of Mathematics, Nanyang Technological University, 469, Bukit Timah Road, Singapore 259756, Republic of Singapore*
E-mail: wongjyp@nievax.nie.ac.sg

*Communicated by Rolf J. Nessel*

Received January 23, 1997; accepted in revised form March 17, 1998

In this paper we offer the best possible error estimates for discrete Abel-Gontscharoff interpolation. As an application, these error bounds are used to provide tests for the right disfocality as well as disconjugacy for higher order difference equations.


## 1. INTRODUCTION

Let $a$, $b$ $(b>a)$ be integers. We shall define the discrete interval $N[a, b] = \{a, a+1, ..., b\}$. The symbol $\Delta^i$ is used to denote the $i$th forward difference operator with stepsize 1. For a nonnegative integer $m$, we define the factorial expression $k^{(m)} = \prod_{\ell=0}^{m-1} (k-\ell)$, with the usual understanding that $k^{(0)} = 1$. Further, for integers $p$, $q$ and any function $u(k)$, we shall denote

$$\oint_{\ell=p}^{q-1} u(\ell) = \begin{cases} \sum_{\ell=p}^{q-1} u(\ell), & \text{if} \quad q \geqslant p \\ -\sum_{\ell=q}^{p-1} u(\ell), & \text{if} \quad p \geqslant q. \end{cases}$$

In this paper we let $u(k)$ be a given function defined on $N[a, b+n-1]$ (with $n \geqslant 2$, $b-a \geqslant n-1$), and let $P_{n-1}(k)$ be the polynomial of degree $(n-1)$ satisfying the *Abel-Gontscharoff interpolating conditions* [2, 3]

$$\Delta^i P_{n-1}(k_{i+1}) = \Delta^i u(k_{i+1}) = A_i, \qquad 0 \leqslant i \leqslant n-1, \tag{1.1}$$

where $k_\ell$, $1 \leqslant \ell \leqslant n$ are integers such that

$$a \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n \leqslant b \qquad (k_n > k_1). \tag{I}$$



0021-9045/99 $30.00

The polynomial $P_{n-1}(k)$ is known as the *Abel-Gontscharoff interpolating polynomial* of $u(k)$. It exists uniquely and can be explicitly expressed as [2, 3]

$$P_{n-1}(k) = \sum_{i=0}^{n-1} T_i(k)\, A_i,$$

where $T_0(k) = 1$ and

$$T_i(k) = \frac{1}{1!\,2!\cdots i!} \begin{vmatrix} 1 & k_1^{(1)} & k_1^{(2)} & \cdots & k_1^{(i-1)} & k_1^{(i)} \\ 0 & 1 & 2k_2^{(1)} & \cdots & (i-1)\,k_2^{(i-2)} & ik_2^{(i-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & (i-1)! & i!\,k_i^{(1)} \\ 1 & k^{(1)} & k^{(2)} & \cdots & k^{(i-1)} & k^{(i)} \end{vmatrix}$$

$$= \oint_{\ell_1=k_1}^{k-1} \oint_{\ell_2=k_2}^{\ell_1-1} \cdots \oint_{\ell_i=k_i}^{\ell_{i-1}-1} 1, \qquad 1 \leqslant i \leqslant n-1.$$

Let $e(k) = u(k) - P_{n-1}(k)$ be the error function associated with the Abel-Gontscharoff interpolation. Our first contribution is the derivation of the *best possible* constants $C_i$, $0 \leqslant i \leqslant n-1$ such that the following error inequalities hold

$$\max_{k \in N[a,\, b+n-1-i]} |\Delta^i e(k)| \leqslant C_i M, \qquad 0 \leqslant i \leqslant n-1, \tag{E}$$

where $M = \max_{\ell \in N[a,\, b-1]} |\Delta^n u(\ell)$. Next, we shall consider the case when $k_1 = a$ and $k_n = b$, i.e., interpolation in the exact sense of the word. Here, the inequalities $(E)$ can be further improved. In fact, assuming that

$$a = k_1 = \cdots = k_{\alpha+1} < k_{\alpha+2} \leqslant \cdots \leqslant k_{n-\beta} \leqslant k_{n-\beta+1} = \cdots = k_n = b, \tag{$I^{\alpha,\beta}$}$$

where $\alpha \in N[0, n-2]$ and $\beta \in N[1, n-1]$ are fixed, we shall obtain the *optimum* constants $C_i$, $0 \leqslant i \leqslant n-1$ in $(E)$. Finally, as an application of the best possible error estimates $(E)$, we shall provide tests for the right disfocality as well as disconjugacy for the difference equation

$$\Delta^n y(k) + p_{n-1}(k)\, \Delta^{n-1} y(k) + \cdots + p_0(k)\, y(k) = 0, \qquad k \in N[a, b-1] \tag{D}$$

in terms of $M_i = \max_{\ell \in N[a,\, b-1]} |p_i(\ell)|$, $0 \leqslant i \leqslant n-1$.

It is noted that the Abel-Gontscharoff conditions (1.1) are quite general and in particular include

(i) the $(m_1, ..., m_r)$ *right focal point conditions*

$$\Delta^i P_{n-1}(k_j) = \Delta^i u(k_j) = A_{i,j},$$
$$1 \leqslant j \leqslant r\ (\geqslant 2),\ a \leqslant k_1 < k_2 < \cdots < k_r \leqslant b, \tag{1.2}$$
$$s_{j-1} \leqslant i \leqslant s_j - 1, \qquad s_0 = 0,\ s_j = \sum_{\ell=1}^{j} m_\ell\ (m_\ell \geqslant 1),\ \sum_{j=1}^{r} m_j = n;$$

and

(ii) the *two-point right focal conditions*

$$\Delta^i P_{n-1}(k_1) = \Delta^i u(k_1) = A_i, \qquad 0 \leqslant i \leqslant \alpha,$$
$$\Delta^i P_{n-1}(k_2) = \Delta^i u(k_2) = B_i, \qquad \alpha + 1 \leqslant i \leqslant n-1, \tag{1.3}$$
$$a \leqslant k_1 < k_2 \leqslant b.$$

The motivation for the present work stems from the continuous Abel-Gontscharoff interpolation [7, 8, 11]. Here, $x(t) \in C^{(n)}[a, b]$ is a given function and $P_{n-1}(t)$ is the Abel-Gontscharoff interpolating polynomial of $x(t)$ satisfying

$$P_{n-1}^{(i)}(a_{i+1}) = x^{(i)}(a_{i+1}), \qquad 0 \leqslant i \leqslant n-1,$$

where $a \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant b$. Levin [9], Coppel [6], Agarwal *et al.* [1, 4, 5], and Wong and Agarwal [12] have bounded the error $e(t) = x(t) - P_{n-1}(t)$ and its derivatives in terms of $\max_{t \in [a,b]} |x^{(n)}(t)|$. Other than extending their work to discrete case, our results also generalize and include the error estimates of Agarwal and Lalli [3] for two-point right focal interpolation (see (1.3)) which they obtained via a different technique, as well as complement several other known right disfocality and disconjugacy tests offered in [5, 6] for differential equations and in [2] for difference equations.

The plan of the paper is as follows. In Section 2 we shall give an error representation in terms of repeated summations. This will be used in Section 3 to establish the best possible error inequalities $(E)$ when $k_\ell$, $1 \leqslant \ell \leqslant n$ satisfy (i) $(I)$; and (ii) $(I^{\alpha,\beta})$. To show the importance of the optimum error estimates obtained, in Section 4 we shall develop tests for the right disfocality as well as disconjugacy for the difference equation $(D)$.

## 2. PRELIMINARIES

THEOREM 2.1. *The error function* $e(k) = u(k) - P_{n-1}(k)$ *associated with the Abel-Gontscharoff interpolation can be written as*

$$e(k) = \oint_{\ell_1=k_1}^{k-1} \oint_{\ell_2=k_2}^{\ell_1-1} \cdots \oint_{\ell_n=k_n}^{\ell_{n-1}-1} \Delta^n u(\ell_n), \qquad k \in N[a, b+n-1]. \tag{2.1}$$

*Proof.* For $0 \leqslant i \leqslant n-1$, the representation (2.1) provides

$$\Delta^i e(k) = \oint_{\ell_1=k_{i+1}}^{k-1} \oint_{\ell_2=k_{i+2}}^{\ell_1-1} \cdots \oint_{\ell_{n-i}=k_n}^{\ell_{n-i-1}-1} \Delta^n u(\ell_{n-i}),$$
$$k \in N[a, b+n-1-i] \tag{2.2}$$

from which it is immediate that $\Delta^i e(k_{i+1}) = 0$, $0 \leqslant i \leqslant n-1$ and $\Delta^n e(k) = \Delta^n u(k)$. ▮

DEFINITION 2.1. Let $u(k)$ be defined on $N[a, b]$. We say that $k = a$ is a *node* of $u(k)$ if $u(a) = 0$. Further, $a < k \leqslant b$ is *node* of $u(k)$ if either $u(k) = 0$ or $u(k-1)\, u(k) < 0$.

Theorem 2.2 (Discrete Rolle's theorem) [2, p. 24]. *Suppose that the function* $u(k)$ *is defined on* $N[a, b]$ *and has* $p$ *nodes, and* $\Delta u(k)$ *is defined on* $N[a, b-1]$ *and has* $q$ *nodes. Then,* $q \geqslant p-1$.

## 3. ERROR ESTIMATES

THEOREM 3.1. *Let* $0 \leqslant i \leqslant n-1$,

$$r_i^{\alpha,\beta} = \begin{cases} \dbinom{n-i-1}{\max\{\alpha-i, \beta, [(n-i-1)/2]\}}, & \text{if } 0 \leqslant i \leqslant n-\beta-1 \\ 1, & \text{if } n-\beta \leqslant i \leqslant n-1 \end{cases} \quad \text{and}$$

$$r_i = r_i^{0,0} = \binom{n-i-1}{\left[\frac{n-i-1}{2}\right]}.$$

(a) *If* $k_\ell$, $1 \leqslant \ell \leqslant n$ *satisfy* $(I)$, *then* $(E)$ *holds with* $C_i = ((b+n-1-a-i)^{(n-i)}/(n-i)!)\, r_i$.

(b) *If* $k_\ell$, $1 \leqslant \ell \leqslant n$ *satisfy* $(I^{\alpha,\beta})$, *then* $(E)$ *holds with* $C_i = ((b+n-1-a-i)^{(n-i)}/(n-i)!)\, r_i^{\alpha,\beta}$.

*Also, the constants* $C_i$, $0 \leqslant i \leqslant n-1$ *are the best possible ones in the respective cases.*

To prove Theorem 3.1, we require the following lemmas.

LEMMA 3.1. *Let* $k_\ell$, $1 \leqslant \ell \leqslant n$ *satisfy* $(I)$. *Then, for each* $0 \leqslant i \leqslant n-1$ *the following holds for* $k \in N[a, k_{i+1}]$,

$$|\Delta^i e(k)| \leqslant M \frac{(b+n-1-k-i)^{(n-i)}}{(n-i)!}. \tag{3.1}$$

*Proof.* Since $k \in N[a, k_{i+1}]$, it follows from (2.2) that

$$\begin{aligned} |\Delta^i e(k)| &\leqslant \sum_{\ell_1=k}^{b-1} \sum_{\ell_2=\ell_1}^{b-1} \cdots \sum_{\ell_{n-i}=\ell_{n-i-1}}^{b-1} |\Delta^n u(\ell_{n-i})| \\ &\leqslant M \frac{(b+n-1-k-i)^{(n-i)}}{(n-i)!}, \qquad 0 \leqslant i \leqslant n-1. \quad \blacksquare \end{aligned}$$

LEMMA 3.2. *Let* $k_\ell$, $1 \leqslant \ell \leqslant n$ *satisfy* $(I^{\alpha,\beta})$. *Then, for each* $n-\beta \leqslant i \leqslant n-1$ *the inequality* (3.1) *holds for* $k \in N[a, b+n-1-i]$.

*Proof.* It is obvious from Lemma 3.1 that (3.1) holds for $k \in N[a, b]$. Now, suppose that $k \in N[b, b+n-1-i]$. Then, in view of (2.2) we find

$$\begin{aligned} |\Delta^i e(k)| &\leqslant \sum_{\ell_1=b}^{k-1} \sum_{\ell_2=b}^{\ell_1-1} \cdots \sum_{\ell_{n-i}=b}^{\ell_{n-i-1}-1} |\Delta^n u(\ell_{n-i})| \\ &\leqslant M \frac{(k-b)^{(n-i)}}{(n-i)!} \leqslant M \frac{(b+n-1-i-b)^{(n-i)}}{(n-i)!} = 0. \end{aligned}$$

This completes the proof of the lemma. ∎

LEMMA 3.3. *Let* $k_\ell$, $1 \leqslant \ell \leqslant n$ *satisfy* $(I^{\alpha,\beta})$. *Then, for each* $0 \leqslant i \leqslant n-\beta-1$ *the following holds for* $k \in N[k_{i+1}, b+n-1-i]$,

$$|\Delta^i e(k)| \leqslant M \frac{(b+n-1-a-i)^{(n-i)}}{(n-i)!} \binom{n-i-1}{\max\left\{\alpha-i, \beta, \left[\frac{n-i-1}{2}\right]\right\}}. \tag{3.2}$$

*Proof.* First, we shall prove that for $k\in N[k_{i+1}, b+n-1-i]$, there exists an integer $j$, $1\leqslant j\leqslant n-i-1$ such that

$$|\Delta^i e(k)|\leqslant M\sum_{\ell_1=a}^{k-1}\sum_{\ell_2=a}^{\ell_1-1}\cdots\sum_{\ell_j=a}^{\ell_{j-1}-1}\sum_{\ell_{j+1}=\ell_j}^{b-1}\sum_{\ell_{j+2}=\ell_{j+1}}^{b-1}\cdots\sum_{\ell_{n-i}=\ell_{n-i-1}}^{b-1}1, \tag{3.3}$$

where $\ell_0=k$. For this, from (2.2) we have

$$\begin{aligned}|\Delta^i e(k)|&\leqslant M\sum_{\ell_1=k_{i+1}}^{k-1}\left|\oint_{\ell_2=k_{i+2}}^{\ell_1-1}\cdots\oint_{\ell_{n-i-1}=k_{n-1}}^{\ell_{n-i-2}-1}\right|\sum_{\ell_{n-i}=\ell_{n-i-1}}^{b-1}1\\&\leqslant M\sum_{\ell_1=a}^{k-1}\left|\oint_{\ell_2=k_{i+2}}^{\ell_1-1}\cdots\oint_{\ell_{n-i-1}=k_{n-1}}^{\ell_{n-i-2}-1}\right|\sum_{\ell_{n-i}=\ell_{n-i-1}}^{b-1}1. \end{aligned}\tag{3.4}$$

If $\ell_1\leqslant k_{i+2}$, then $|\oint_{\ell_2=k_{i+2}}^{l_1-1}|=|-\sum_{\ell_2=\ell_1}^{k_{i+2}-1}|\leqslant\sum_{\ell_2=\ell_1}^{b-1}$. Further, since $\ell_1\leqslant\ell_2\leqslant k_{i+2}-1\leqslant k_{i+3}-1$, we have $|\oint_{\ell_3=k_{i+3}}^{\ell_2-1}|=|-\sum_{\ell_3=\ell_2}^{k_{i+3}-1}|\leqslant\sum_{\ell_3=\ell_2}^{b-1}$. Continuing in this way, (3.4) leads to (3.3) with $j=1$.

If $\ell_1\geqslant k_{i+2}$, then $|\oint_{\ell_2=k_{i+2}}^{\ell_1-1}|=\sum_{\ell_2=k_{i+2}}^{\ell_1-1}\leqslant\sum_{\ell_2=a}^{\ell_1-1}$. Noting that $k_{i+2}\leqslant\ell_2\leqslant\ell_1-1$, there are now two possibilities:

*Case* 1. $\ell_2\leqslant k_{i+3}$. We find that $|\oint_{\ell_3=k_{i+3}}^{\ell_2-1}|=|-\sum_{\ell_3=\ell_2}^{k_{i+3}-1}|\leqslant\sum_{\ell_3=\ell_2}^{b-1}$. By using a previous argument, it follows that $|\oint_{\ell_4=k_{i+4}}^{\ell_3-1}|\leqslant\sum_{\ell_4=\ell_3}^{b-1}$, and so on. Hence, from (3.4) we get (3.3) with $j=2$.

*Case* 2. $\ell_2\geqslant k_{i+3}$. Hence, since $|\oint_{\ell_3=k_{i+3}}^{\ell_2-1}|=\sum_{\ell_3=k_{i+3}}^{\ell_2-1}\leqslant\sum_{\ell_3=a}^{\ell_2-1}$, it follows from (3.4) that

$$|\Delta^i e(k)|\leqslant M\sum_{\ell_1=a}^{k-1}\sum_{\ell_2=a}^{\ell_1-1}\sum_{\ell_3=a}^{\ell_2-1}\left|\oint_{\ell_4=k_{i+4}}^{\ell_3-1}\cdots\oint_{\ell_{n-i-1}=k_{n-1}}^{\ell_{n-i-2}-1}\right|\sum_{\ell_{n-i}=\ell_{n-i-1}}^{b-1}1. \tag{3.5}$$

Noting that $k_{i+3}\leqslant\ell_3\leqslant\ell_2-1$, once again we have two subcases, either $\ell_3\leqslant k_{i+4}$, in such a situation (3.5) leads to (3.3) with $j=3$; or $\ell_3\geqslant k_{i+4}$, for which (3.5) provides

$$|\Delta^i e(k)|\leqslant M\sum_{\ell_1=a}^{k-1}\sum_{\ell_2=a}^{\ell_1-1}\sum_{\ell_3=a}^{\ell_2-1}\sum_{\ell_4=a}^{\ell_3-1}\left|\oint_{\ell_5=k_{i+5}}^{\ell_4-1}\cdots\oint_{\ell_{n-i-1}=k_{n-1}}^{\ell_{n-i-2}-1}\right|\sum_{\ell_{n-i}=\ell_{n-i-1}}^{b-1}1.$$

Continuing the process, we see that (3.3) holds for some $j\in N[1, n-i-1]$.

Next, noting that the right side of (3.3) attains its maximum when $k=b+n-1-i$, we evaluate the right side of (3.3) to get

$$
\begin{aligned}
|\Delta^i e(k)| &\leqslant \frac{M}{(n-i)!}\Bigg\{(-1)^j\,(b+n-1-k-i)^{(n-i)}+\sum_{\tau=0}^{j-1}(-1)^{j-\tau+1}\binom{n-i}{\tau}\\
&\qquad\times(b+n-1-a-i-\tau)^{(n-i-\tau)}\,(k-a)^{(\tau)}\Bigg\}\\
&\leqslant \frac{M}{(n-i)!}\sum_{\tau=0}^{j-1}(-1)^{j-\tau+1}\binom{n-i}{\tau}\\
&\qquad\times(b+n-1-a-i-\tau)^{(n-i-\tau)}\,(b+n-1-a-i)^{(\tau)}\\
&=\frac{M}{(n-i)!}\sum_{\tau=0}^{j-1}(-1)^{j-\tau+1}\binom{n-i}{\tau}(b+n-1-a-i)^{(n-i)}\\
&=\frac{M}{(n-i)!}\binom{n-i-1}{j-1}(b+n-1-a-i)^{(n-i)},\\
&\qquad k\in N[k_{i+1},\, b+n-1-i],
\end{aligned}
\tag{3.6}
$$

where a known identity [10, p. 53] is used in the last equality.

Now, we shall maximize the right side of (3.6) over $j$. For this, if $0\leqslant i\leqslant\alpha$ $(\leqslant n-\beta-1)$, then it is clear from (2.2) and (3.3) that we must have $j\geqslant\alpha-i+1$ or $j-1\geqslant\alpha-i$. Moreover, since $i\leqslant n-\beta-1$, from (2.2) and (3.3) again we observe that $n-i-j\geqslant\beta$. Coupling all these and noting the relation $\binom{n-i-1}{j-1}=\binom{n-i-1}{n-i-j}$ and also using the properties of binomial coefficients, we see that

$$
\max_j\binom{n-i-1}{j-1}=\max_{\substack{\ell\geqslant\alpha-i\\ \ell\geqslant\beta}}\binom{n-i-1}{\ell}
=\binom{n-i-1}{\max\left\{\alpha-i,\,\beta,\left[\dfrac{n-i-1}{2}\right]\right\}}.
$$

Hence, from (3.6) we immediately get (3.2). ∎

*Proof of Theorem* 3.1(a). Since the function $g(k)=\Delta^i e(k)$ satisfies $g(k_{i+1})=\Delta g(k_{i+2})=\cdots=\Delta^{n-i-1}g(k_n)=0$, $\Delta^{n-i}g(k)=\Delta^n u(k)$, and $a\leqslant k_{i+1}\leqslant\cdots\leqslant k_n\leqslant b$, it suffices to prove $(E)$ for $i=0$ only.

First, we note that for $k\in N[a,k_1]$, (3.1) implies that (when $i=0$)

$$
|e(k)|\leqslant M\,\frac{(b+n-1-a)^{(n)}}{n!}. \tag{3.7}
$$

Next, it is clear that

$$e(k)=\sum_{\ell_1=k_1}^{k-1}\sum_{\ell_2=k_2}^{\ell_1-1}\cdots\sum_{\ell_i=k_i}^{\ell_{i-1}-1}\Delta^i e(\ell_i),\qquad 1\leqslant i\leqslant n, \tag{3.8}$$

where $\ell_0=k$. So for $k\geqslant k_i$, $1\leqslant i\leqslant n$ it follows from (3.8) that

$$\begin{aligned}|e(k)|&\leqslant\sum_{\ell_1=a}^{k-1}\sum_{\ell_2=a}^{\ell_1-1}\cdots\sum_{\ell_i=a}^{\ell_{i-1}-1}|\Delta^i e(\ell_i)|\\&=\sum_{\ell_i=a}^{k-1}\sum_{\ell_{i-1}=\ell_i+1}^{k-1}\cdots\sum_{\ell_2=\ell_3+1}^{k-1}\sum_{\ell_1=\ell_2+1}^{k-1}|\Delta^i e(\ell_i)|\\&=\sum_{\ell_i=a}^{k-1}|\Delta^i e(\ell_i)|\,\frac{(k-\ell_i-1)^{(i-1)}}{(i-1)!}.\end{aligned}\tag{3.9}$$

In particular, for $k\in N[k_n, b+n-1]$, (3.9) leads to (when $i=n$)

$$\begin{aligned}|e(k)|&\leqslant M\sum_{\ell_n=a}^{b+n-1-1}\frac{(b+n-1-\ell_n-1)^{(n-1)}}{(n-1)!}\\&=M\,\frac{(b+n-1-a)^{(n)}}{n!}.\end{aligned}\tag{3.10}$$

Coupling (3.1) and (3.9), it follows that for $k\in N[k_i, k_{i+1}]$, $1\leqslant i\leqslant n-1$,

$$\begin{aligned}|e(k)|&\leqslant M\sum_{\ell_i=a}^{k-1}\frac{(b+n-1-\ell_i-i)^{(n-i)}}{(n-i)!}\,\frac{(k-\ell_i-1)^{(i-1)}}{(i-1)!}\\&\leqslant M\sum_{\ell_i=a}^{b-1}\frac{(b+n-1-\ell_i-i)^{(n-i)}}{(n-i)!}\,\frac{(b-\ell_i-1)^{(i-1)}}{(i-1)!}\\&=M\sum_{\ell_i=a}^{b-1}\frac{(b+n-1-\ell_i-i)^{(n-1)}}{(n-i)!\,(i-1)!}\\&=\frac{M}{(n-i)!\,(i-1)!}\sum_{\ell=n-i}^{b+n-1-a-i}\ell^{(n-1)}\\&\leqslant\frac{M}{(n-i)!\,(i-1)!}\sum_{\ell=1}^{b+n-2-a}\ell^{(n-1)}\\&=M\binom{n-1}{n-i}\frac{(b+n-1-a)^{(n)}}{n!}\\&\leqslant M\binom{n-1}{[(n-1)/2]}\frac{(b+n-1-a)^{(n)}}{n!},\end{aligned}\tag{3.11}$$

where in the last inequality we have used the fact that for $1 \leqslant i \leqslant n-1$, the binomial coefficient $\binom{n-1}{n-i}$ attains its maximum when $n-i=[(n-1)/2]$.

Now, a combination of (3.7), (3.10), and (3.11) gives $(E)$ for $i=0$ immediately. ∎

*Proof of Theorem* 3.1(b). From Lemma 3.2, $(E)$ is immediate for each $n-\beta \leqslant i \leqslant n-1$. Next, for $0 \leqslant i \leqslant n-\beta-1$, we combine Lemmas 3.1, 3.3 and the fact that any binomial coefficient is at least 1 to obtain $(E)$. ∎

Theorem 3.1(b) leads to the following corollaries.

COROLLARY 3.1. *For the* $(m_1, ..., m_r)$ *right focal point interpolation* (1.2) *with* $k_1=a$ *and* $k_r=b$, *the following inequalities hold*

$$\max_{k \in N[a, b+n-1-i]} |\Delta^i e(k)| \leqslant M \frac{(b+n-1-a-i)^{(n-i)}}{(n-i)!} r_i^{m_1-1, m_r}, \qquad 0 \leqslant i \leqslant n-1.$$

*Proof.* It is noted in this case that $\alpha = m_1 - 1$ and $\beta = m_r$. ∎

COROLLARY 3.2. *For the two-point right focal interpolation* (1.3) *with* $k_1=a$ *and* $k_2=b$, *the following inequalities hold*

$$\begin{aligned}\max_{k \in N[a, b+n-1-i]} &|\Delta^i e(k)| \\ &\leqslant M \frac{(b+n-1-a-i)^{(n-i)}}{(n-i)!} r_i^{\alpha, n-\alpha-1}, \qquad 0 \leqslant i \leqslant n-1 \\ &= M \frac{(b+n-1-a-i)^{(n-i)}}{(n-i)!} \begin{cases} \binom{n-i-1}{\alpha-i}, & \text{if } 0 \leqslant i \leqslant \alpha \\ 1, & \text{if } \alpha+1 \leqslant i \leqslant n-1. \end{cases}\end{aligned}$$

*Proof.* Here, we have $\alpha+1+\beta=n$, i.e., $\beta=n-\alpha-1$. Since $\binom{n-i-1}{\alpha-i} = \binom{n-i-1}{n-\alpha-1} = \binom{n-i-1}{\beta}$, from the symmetrical property of binomial coefficients it is clear that

$$\binom{n-i-1}{\max\left\{\alpha-i, \beta, \left[\frac{n-i-1}{2}\right]\right\}} = \binom{n-i-1}{\alpha-i}. \quad ∎$$

*Remark* 3.1. Corollary 3.2 has also been obtained by Agarwal and Lalli [3, Theorem 7.4], however, by using a different error representation which is in terms of Green's function.

*Remark* 3.2. Theorem 3.1(b) offers the best possible error inequalities $(E)$. To prove this, for a fixed $i$, $0 \leqslant i \leqslant n-1$ we define the $n$th degree polynomial

$$u_\theta(k) = \frac{(-1)^n}{n!} \sum_{\tau=\theta}^{n} (-1)^\tau \binom{n}{\tau} (k-a)^{(\tau)} (b+n-1-a-\tau)^{(n-\tau)}, \quad k \in N[a, b+n-1], \tag{3.12}$$

where

$$\theta = \theta(i) = \begin{cases} n-\beta, & \text{if } \beta > \max\{\alpha - i, [(n-i-1)/2]\} \\ \max\{\alpha - i, \beta, [(n-i-1)/2]\} + i + 1, & \text{otherwise.} \end{cases} \tag{3.13}$$

Obviously, $\Delta^n u_\theta(k) = (-1)^{2n} = 1 = \max_{k \in N[a, b-1]} |\Delta^n u_\theta(k)| = M$. Further, we have for $0 \leqslant j \leqslant n-1$ and $k \in N[a, b+n-1-j]$,

$$\Delta^j u_\theta(k) = (-1)^n \frac{j!}{n!} \sum_{\tau=\max\{\theta, j\}}^{n} (-1)^\tau \binom{n}{\tau}\binom{\tau}{j} \times (k-a)^{(\tau-j)} (b+n-1-a-\tau)^{(n-\tau)}. \tag{3.14}$$

It follows from (3.14) that

$$\Delta^j u_\theta(a) = 0, \qquad 0 \leqslant j \leqslant \theta - 1. \tag{3.15}$$

Also, we claim that

$$\Delta^j u_\theta(b) = 0, \qquad \theta \leqslant j \leqslant n-1. \tag{3.16}$$

In fact, from (3.14) it can easily be checked that (3.16) holds for $j = n-1$. Further, for $\theta \leqslant j \leqslant n-2$, we find that

$$\begin{aligned} \Delta^j u_\theta(b) &= (-1)^n \frac{j!}{n!} \sum_{\tau=j}^{n} (-1)^\tau \binom{n}{\tau}\binom{\tau}{j} (b-a)^{(\tau-j)} (b+n-1-a-\tau)^{(n-\tau)} \\ &= (-1)^n \frac{j!}{n!} (b-a) \sum_{\tau=j}^{n} (-1)^\tau \binom{n}{\tau}\binom{\tau}{j} (b+n-1-a-\tau)^{(n-1-j)} \end{aligned}$$

$$= (-1)^n \frac{j!}{n!} (b-a) \sum_{\ell=0}^{n-j} (-1)^{\ell+j} \binom{n}{\ell+j} \binom{\ell+j}{j}$$

$$\times (b+n-1-a-j-\ell)^{(n-1-j)}$$

$$= (-1)^{n+j} \frac{j!}{n!} (b-a) \binom{n}{j} \sum_{\ell=0}^{n-j} (-1)^{\ell} \binom{n-j}{\ell}$$

$$\times (b+n-1-a-j-\ell)^{(n-1-j)}$$

$$= (-1)^{n+j} \frac{j!}{n!} (b-a) \binom{n}{j} (n-j-1)!$$

$$\times \sum_{\ell=0}^{n-j} (-1)^{\ell} \binom{n-j}{\ell} \binom{b+n-1-a-j-\ell}{n-j-1}$$

$$= (-1)^{n+j} \frac{j!}{n!} (b-a) \binom{n}{j} (n-j-1)!$$

$$\times \sum_{\ell=0}^{n-j} (-1)^{\ell} \binom{n-j}{\ell} \binom{b+n-1-a-j-\ell}{b-a-\ell}$$

$$= (-1)^{n+j} \frac{j!}{n!} (b-a) \binom{n}{j} (n-j-1)! \binom{b-a-1}{b-a} = 0,$$

where an identity of [10, p. 8] has been used in the second last equality. This completes the proof of (3.16).

Let $P_{n-1}(k)$ be the Abel-Gontscharoff interpolating polynomial of $u_\theta(k)$ satisfying the following interpolating conditions

$$\Delta^j P_{n-1}(a) = \Delta^j u_\theta(a), \qquad 0 \leqslant j \leqslant \theta - 1;$$

$$\Delta^j P_{n-1}(b) = \Delta^j u_\theta(b), \qquad \theta \leqslant j \leqslant n-1.$$

Then, in view of (3.15) and (3.16), we see that $P_{n-1}(k) \equiv 0$ and hence $e(k) = u_\theta(k)$.

If $\theta \leqslant i \leqslant n-1$, then it follows from (3.14) that

$$\max_{k \in N[a, b+n-1-i]} |\Delta^i u_\theta(k)| \geqslant |\Delta^i u_\theta(a)| = \frac{1}{(n-i)!} (b+n-1-a-i)^{(n-i)}. \tag{3.17}$$

If $0 \leqslant i \leqslant \theta - 1$, then from (3.14) we find that

$$\begin{aligned}
\max_{k \in N[a,\, b+n-1-i]} |\Delta^i u_\theta(k)| &\geqslant |\Delta^i u_\theta(b+n-1-i)| \\
&= \frac{i!}{n!} \left| \sum_{\tau=\theta}^{n} (-1)^\tau \binom{n}{\tau}\binom{\tau}{i} \right| (b+n-1-a-i)^{(n-i)} \\
&= \frac{1}{(n-i)!} \left| \sum_{\ell=0}^{n-\theta} (-1)^{n-\ell} \binom{n-i}{\ell} \right| (b+n-1-a-i)^{(n-i)} \\
&= \frac{1}{(n-i)!} \binom{n-i-1}{n-\theta} (b+n-1-a-i)^{(n-i)} \\
&= \frac{1}{(n-i)!} \binom{n-i-1}{\max\left\{\alpha - i,\, \beta,\, \left[\frac{n-i-1}{2}\right]\right\}} (b+n-1-a-i)^{(n-i)},
\end{aligned} \tag{3.18}$$

where we have used an identity of [10, p. 53] and the definition of $\theta$, respectively, in the last two equalities. Subsequently, a combination of (3.17) and (3.18) yields

$$\begin{aligned}
\max_{k \in N[a,\, b+n-1-i]} |\Delta^i u_\theta(k)| &\geqslant \frac{(b+n-1-a-i)^{(n-i)}}{(n-i)!} \\
&\quad \times \begin{cases} \binom{n-i-1}{\max\{\alpha - i,\, \beta,\, [(n-i-1)/2]\}}, & \text{if} \quad 0 \leqslant i \leqslant \theta - 1 \\ 1, & \text{if} \quad \theta \leqslant i \leqslant n-1. \end{cases}
\end{aligned} \tag{3.19}$$

We shall now show that

(i) if $0 \leqslant i \leqslant n - \beta - 1$, then $i \leqslant \theta - 1$; and

(ii) if $n - \beta \leqslant i \leqslant n - 1$, then $\theta = n - \beta$.

To justify (i), there are two cases to consider.

*Case* 1. $\theta = n - \beta$. In this case, $i \leqslant n - \beta - 1$ means $i \leqslant \theta - 1$.

*Case* 2. $\theta = \max\{\alpha - i, \beta, [(n-i-1)/2]\} + i + 1 = [(n+i+1)/2]$ or $(\alpha+1)$. First, let $\theta = [(n+i+1)/2]$. Then, $i \leqslant \theta - 1$ is the same as $i \leqslant n-1$ when $(n+i)$ is odd (which is obviously true), and is equivalent to $i \leqslant n-2$ when $(n+i)$ is even (which is true because $i \leqslant n-\beta-1 \leqslant n-2$). Next, suppose that $\theta = \alpha+1$. Here, we have $\max\{\alpha - i, \beta, [(n-i-1)/2]\} = \alpha - i \geqslant 0$ or $i \leqslant \alpha = \theta - 1$.

Next, to prove (ii) it is sufficient to show that $\beta > \max\{\alpha - i, [(n-i-1)/2]\}$. For this, it is noted that $\alpha - i \leqslant \alpha - (n-\beta) < 0$. Further, we have $[(n-i-1)/2] < \beta$ as this is equivalent to $i > n - 2\beta - 1$ when $(n-i)$ is odd (which is true), and is the same as $i > n-2\beta-2$ when $(n-i)$ is even (which is true). Hence, $\beta > \max\{\alpha - i, [(n-i-1)/2]\}$.

Finally, in view of (i) and (ii), (3.19) subsequently leads to

$$\max_{k \in N[a, b+n-1-i]} |\Delta^i u_\theta(k)| \geqslant \frac{(b+n-1-a-i)^{(n-i)}}{(n-i)!} r_i^{\alpha, \beta}. \tag{3.20}$$

Hence, for the function $u_\theta(k)(=e(k))$ equality holds in $(E)$. This shows that the error inequalities $(E)$ are the best possible.

*Remark* 3.3. Theorem 3.1(a) provides the best possible error inequalities $(E)$. Note that in this case the constants $C_i$ obtained are actually those of Theorem 3.1(b) when $\alpha = \beta = 0$. Therefore, from (3.13) we have $\theta = \theta(i) = [(n-i-1)/2] + i + 1 = [(n+i+1)/2]$ and the verification is similar to that in Remark 3.2.

*Remark* 3.4. For each $0 \leqslant i \leqslant n-1$, $r_i^{\alpha, \beta} \leqslant r_i$. This is obvious as

$$\binom{n-i-1}{\left[\frac{n-i-1}{2}\right]} \geqslant \binom{n-i-1}{\max\left\{\alpha - i, \beta, \left[\frac{n-i-1}{2}\right]\right\}}, \qquad \text{if} \quad 0 \leqslant i \leqslant n-\beta-1 \tag{3.21}$$

$$\geqslant 1, \qquad \text{if} \quad n-\beta \leqslant i \leqslant n-1. \tag{3.22}$$

In fact, we have strict inequality in (3.21) if

$$\left[\frac{n-i-1}{2}\right] < \begin{cases} \max\{\alpha - i, \beta\}, & \text{if} \quad (n-i-1) \text{ is even} \\ \max\{\alpha - i, \beta\} - 1, & \text{if} \quad (n-i-1) \text{ is odd.} \end{cases}$$

Further, in (3.22) the strict inequality holds provided $[(n-i-1)/2] > 0$ or $n-\beta \leqslant i \leqslant n-3$.

## 4. TESTS FOR RIGHT DISFOCALITY AND DISCONJUGACY

To illustrate the importance of the error inequalities obtained in Section 3, we shall provide tests for the right disfocality as well as disconjugacy for the difference equation $(D)$.

DEFINITION 4.1. The difference equation $(D)$ is said to be $(\alpha, \beta)$-*right disfocal* on $N[a, b+n-1]$ if and only if the only solution of $(D)$ satisfying

$$\Delta^i y(k_{i+1}) = 0, \qquad 0 \leqslant i \leqslant n-1, \tag{4.1}$$

where $k_\ell$, $1 \leqslant \ell \leqslant n$ fulfill $(I^{\alpha, \beta})$, is the trivial solution. Further, we say that $(D)$ is *right disfocal* on $N[a, b+n-1]$ if and only if the only solution of $(D)$ satisfying (4.1) where $k_\ell$, $1 \leqslant \ell \leqslant n$ fulfill $(I)$, is the trivial solution.

THEOREM 4.1. *Let*

$$\phi^{\alpha, \beta}(h) = \sum_{i=0}^{n-1} \frac{M_i (h-i)^{(n-i)}}{(n-i)!} r_i^{\alpha, \beta} \qquad \textit{and} \qquad \phi = \phi^{0, 0}.$$

(a) *If* $\phi^{\alpha, \beta}(b+n-1-a) \leqslant 1$, *then* $(D)$ *is* $(\alpha, \beta)$-*right disfocal on* $N[a, b+n-1]$.

(b) *If* $\phi(b+n-1-a) \leqslant 1$, *then* $(D)$ *is right disfocal on* $N[a, b+n-1]$.

*Proof.* (a) Suppose on the contrary that $(D)$ has a nontrivial solution $y(k)$ satisfying (4.1) where $k_\ell$, $1 \leqslant \ell \leqslant n$ fulfill $(I^{\alpha, \beta})$. Then, the Abel-Gontscharoff interpolating polynomial $P_{n-1}(k)$ of $y(k)$ is zero and so $e(k) = y(k) - P_{n-1}(k) = y(k)$. Applying Theorem 3.1(b) we obtain

$$\max_{k \in N[a, b+n-1-i]} |\Delta^i y(k)| \leqslant M \frac{(b+n-1-a-i)^{(n-i)}}{(n-i)!} r_i^{\alpha, \beta}, \qquad 0 \leqslant i \leqslant n-1, \tag{4.2}$$

where $M = \max_{k \in N[a, b-1]} |\Delta^n y(k)| = |\Delta^n y(\tau)|$ for some $\tau \in N[a, b-1]$. Subsequently, using (4.2) we find that

$$M = |\Delta^n y(\tau)| = |p_0(\tau)\, y(\tau) + \cdots + p_{n-1}(\tau)\, \Delta^{n-1} y(\tau)| \tag{4.3}$$

$$\leqslant \sum_{i=0}^{n-1} M_i\, |\Delta^i y(\tau)|$$

$$\leqslant M \phi^{\alpha, \beta}(b+n-1-a). \tag{4.4}$$

Clearly, $M>0$, since otherwise $y(k)$ would coincide on $N[a, b+n-1]$ with a polynomial of degree $m<n$ and $\Delta^m y(k)$ would not vanish on $N[a, b+n-1-m]$. Hence, it follows from (4.4) that $\phi^{\alpha,\beta}(b+n-1-a)\geqslant 1$. It only remains to exclude the possibility of equality. At least one of the numbers $M_i$, $0\leqslant i\leqslant n-1$ is different from zero, since otherwise $y(k)$ would be a polynomial of degree less than $n$ and cannot satisfy (4.1). Thus, if $\phi^{\alpha,\beta}(b+n-1-a)=1$ then equality must hold in (4.2) for at least one value of $i$. In view of Remark 3.2 this is possible only if $y(k)$ coincides on $N[a, b+n-1]$ with a polynomial of degree $n$. But we can then take $\tau$ to be any point on $N[a, b-1]$, and $|\Delta^i y(\tau)|$ is not constant on $N[a, b+n-1-i]$ for any $0\leqslant i\leqslant n-1$. So (4.3) cannot hold. Therefore, we must have $\phi^{\alpha,\beta}(b+n-1-a)>1$.

(b) The proof is similar to that of Case (a) with the obvious modification that we employ Theorem 3.1(a) and Remark 3.3 in the arguments. ∎

DEFINITION 4.2. The difference equation $(D)$ is said to be *disconjugate* on $N[a, b+n-1]$ if no nontrivial solution of $(D)$ has $n$ nodes on $N[a, b+n-1]$.

COROLLARY 4.1. *If $\phi(b+n-1-a)\leqslant 1$, then $(D)$ is disconjugate on $N[a, b+n-1]$.*

*Proof.* By Theorem 2.2, the right disfocality of $(D)$ on $N[a, b+n-1]$ implies the disconjugacy of $(D)$ on $N[a, b+n-1]$. The conclusion now follows immediately from Theorem 4.1(b). ∎

LEMMA 4.1. *The difference equation $(D)$ is disconjugate on $N[a, b+n-1]$ if and only if for any $r$ distinct integers $a\leqslant k_1<\cdots<k_r\leqslant b$ and for any $n$ arbitrary constants $A_{i,j}$ where $1\leqslant j\leqslant r$, $0\leqslant i\leqslant m_j$ and $\sum_{j=1}^{r} m_j+r=n$, there exists a solution $y(k)$ of $(D)$ such that*

$$\Delta^i y(k_j)=A_{i,j}. \tag{4.5}$$

*Proof.* Let $y_i(k)$, $1\leqslant i\leqslant n$ be linearly independent solutions of $(D)$. Then, any solution of $(D)$ has the form

$$y(k)=\sum_{i=1}^{n} b_i y_i(k),$$

where $b_i$, $1\leqslant i\leqslant n$ are some constants. Writing $B=[b_i]$,

$$A=[A_{0,1}, ..., A_{m_1,1}, A_{0,2}, ..., A_{m_2,2}, ..., A_{0,r}, ..., A_{m_r,r}]^T$$

and

$$Y=\begin{bmatrix} y_1(k_1) & \cdots & y_n(k_1) \\ \cdots & \cdots & \cdots \\ \Delta^{m_1}y_1(k_1) & \cdots & \Delta^{m_1}y_n(k_1) \\ y_1(k_2) & \cdots & y_n(k_2) \\ \cdots & \cdots & \cdots \\ \Delta^{m_2}y_1(k_2) & \cdots & \Delta^{m_2}y_n(k_2) \\ \cdots & \cdots & \cdots \\ y_1(k_r) & \cdots & y_n(k_r) \\ \cdots & \cdots & \cdots \\ \Delta^{m_r}y_1(k_r) & \cdots & \Delta^{m_r}y_n(k_r) \end{bmatrix},$$

we want to choose $B$ such that $YB = A$. By linear algebra, this is possible for every $A$ if and only if the homogeneous system $YB = 0$ has only the trivial solution, which is exactly the case as $(D)$ is disconjugate on $N[a, b+n-1]$. ∎

*Remark* 4.1. It follows from Lemma 4.1 that disconjugacy of a difference equation means the possibility of interpolation by the solutions of the difference equation.

COROLLARY 4.2. *If $\phi(b+n-1-a) \leqslant 1$, then the boundary value problem $(D)$, (4.5) has a unique solution on $N[a, b+n-1]$.*

*Proof.* This is an immediate consequence of Corollary 4.1 and Lemma 4.1. ∎

## ACKNOWLEDGMENTS

The author is grateful to referees for their comments on the first draft of the paper.

## REFERENCES

1. R. P. Agarwal, "Boundary Value Problems for Higher Order Differential Equations," World Scientific, Singapore, 1986.
2. R. P. Agarwal, "Difference Equations and Inequalities," Dekker, New York, 1992.
3. R. P. Agarwal and B. S. Lalli, Discrete polynomial interpolation, Green's functions, maximum principles, error bounds and boundary value problems, *Comput. Math. Appl.* **25** (1993), 3–39.
4. R. P. Agarwal and R. A. Usmani, Iterative methods for solving right focal point boundary value problems, *J. Comput. Appl. Math.* **14** (1986), 371–390.

5. R. P. Agarwal and P. J. Y. Wong, "Error Inequalities in Polynomial Interpolation and Their Applications," Kluwer Academic, Dordrecht, 1993.
6. W. A. Coppel, "Disconjugacy," Lecture Notes in Mathematics, Vol. 220, Springer-Verlag, New York, 1971.
7. P. J. Davis, "Interpolation and Approximation," Blaisdell, Boston, 1961.
8. V. L. Gontscharoff, "Theory of Interpolation and Approximation of Functions," Gostekhizdat, Moscow, 1954.
9. A. J. Levin, A bound for a function with monotonely distributed zeros of successive derivatives, *Mat. Sb.* **64** (106) (1964), 396–409.
10. J. Riordan, "Combinatorial Identities," Wiley, New York, 1968.
11. J. M. Whittaker, "Interpolatory Function Theory," Cambridge, 1935.
12. P. J. Y. Wong and R. P. Agarwal, Abel-Gontscharoff interpolation error bounds for derivatives, *Proc. Roy. Soc. Edinburgh Sect. A* **119** (1991), 367–372.